Douglas C. Wadle

# *Introduction to Reframing:*

## *augmentation and diminution of extended just intonation frequency ratios*[1]

### 1.0. Introduction

This paper explains the mechanics of “reframing” (intervallic augmentation and diminution of extended just intonation frequency ratios), introduces some basic musical operations on reframed ratio sets, and presents a tripartite model for describing the ratio sets involved in reframing. Reframing occurs in an open context, by which I mean that there is no predetermined limiting factor as to which pitches or pitch classes may appear. In other words, in the augmentation or diminutions of frequency ratios, one is not primarily concerned with a preexisting, closed set of pitches comprising a scale, a closed subset of pitches (scale) chosen from a closed set of pitches comprising a tuning system, or even a closed set of pitches comprising a tuning system. This is a significant difference from such traditional musical theoretical approaches to pitch transformation as pitch-class set theory and neo-Riemannian theory, both of which treat pitch classes as integers modulo 12. By contrast, under reframing a new frequency ratio, indicating a pitch or pitch interval, which was not found within the source material (that which is reframed) or in some closed set of available pitches or pitch-classes, may well be introduced. The paper is concerned specifically with the reframing operation and assumes a working knowledge of

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frequency ratios as a musical resource, though a brief review of basic transposition and inversion of frequency ratios is included (section 2.1.).

**2.0. Introducing Reframing: Terms and Equations**

The process described herein was first conceived as the projection of some musical interval, expressed as a frequency ratio, onto a bounding interval, or "frame" (hence "reframing"), to be denoted by "*f*". In other words, the goal was simply to divide the span of some interval other than the octave according to the same proportions as an existing division of the span of an octave occurring in a bit of music in just intonation. In the course of compositional explorations[2] of reframing, some limitations of this conception were noted (and were, in some cases, exploited for aesthetic ends), resulting in a re-description of the process as it appears below.

The reframing of a frequency ratio is accomplished through a simple linear function that can be represented graphically by plotting the original frequency ratios along the x-axis and the resulting frequency ratios along the y-axis (see Fig. 1). The graph of the resulting line has a slope equal to $f - 1$ and intersects the point ($^1/_1$, $^1/_1$). So, for the $^2/_1$ frame, the slope is $^1/_1$ and all resulting ratios map directly onto the original ratios (making this the identity function) whereas, for the $^5/_2$ frame, the slope will be $^3/_2$ and the resulting ratios will be altered to fall along this slope. Frames less than two will result in diminution of frequency ratios. A frame greater than two will result in augmentation. A frame less than one gives a negative slope, meaning that

[2] See (and hear) Douglas C. Wadle, *Cognitive Disjunction* (2006), *Cognitive Congruence* (2007), *Cloister Walk, No. 2* (2007), "Systema"(2008), "Systema (with linear distortion)" (2008), "After 'Ohio Impromptu'" (2009), and "Systema II" (2010).

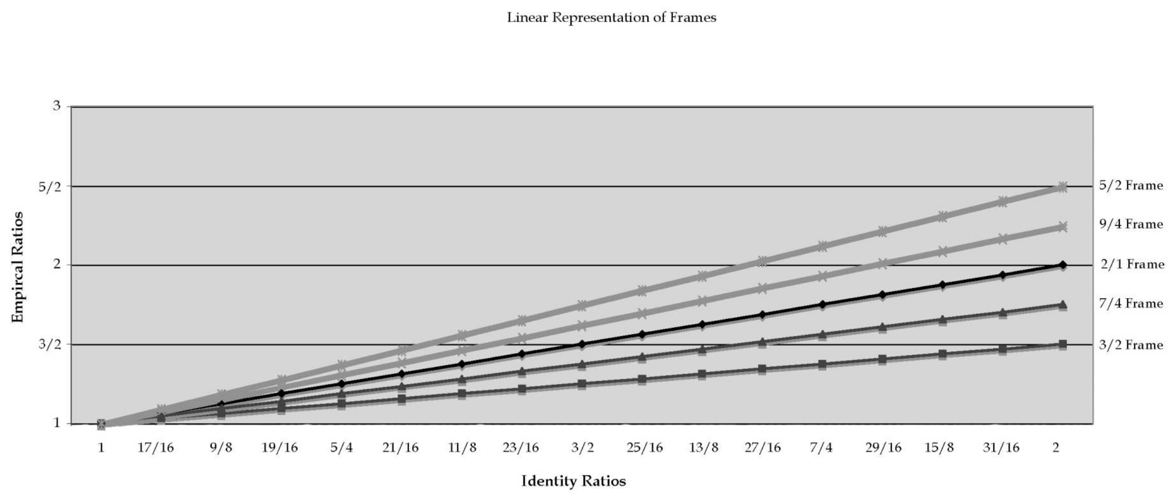
successive ratios increasingly greater than $^1/_1$ in the original ratio set will result in (diminished) ratios successively less than $^1/_1$ in the reframed set.

The introduction of some terminology should help maintain clarity in the following discussion. In addition to the "frame", we have an interval that expresses the original pitch interval to be reframed, which I refer to as the abstract ratio, or abstraction (to be denoted by "*a*"). The ratio resulting from the process is termed the "empirical ratio" (denoted by "*e*"). This language allows us to state, "The empirical ratio of the *a*-abstraction at the *f*-frame is *e*". It also allows us to keep track of abstract ratios across frames, as it may be useful to conceive *e* as the (empirical ratio of the) *a* (abstraction) of (the) *f* (frame) in comparison with various empirical ratios of the *a*-abstraction in other frames. The function for determining the empirical ratio, *e*, at a given frame, *f*, is:

$$g_f(a) \quad = \quad [(a - 1)(f - 1)] + 1 \quad .$$

When the resulting values, *e*, are plotted against the corresponding values of *a*, we get the line of slope $f - 1$ discussed above (see Fig. 1). The range of the function is equal to the set of *a* values, the domain the set of values for *e*[3].

[3] The general function for reframing is denoted "$g(f, a)$". The denotation "$g_f(a)$" gives a range of reframing functions that can be specified by the value of *f*, in keeping with the spirit of the terminology introduced above.




**Empirical Ratios:**

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| At the 5/2 Frame | 1 | 35/32 | 19/16 | 41/32 | 11/8 | 47/32 | 25/16 | 53/32 | 7/4 | 59/32 | 31/16 | 65/32 | 17/8 | 71/32 | 37/16 | 77/32 | 5/2 |
| At the 9/4 Frame | 1 | 69/64 | 37/32 | 79/64 | 21/16 | 89/64 | 47/32 | 99/64 | 13/8 | 109/64 | 57/32 | 119/64 | 31/16 | 129/64 | 67/32 | 139/64 | 9/4 |
| At the 2/1 Frame | 1 | 17/16 | 9/8 | 19/16 | 5/4 | 21/16 | 11/8 | 23/16 | 3/2 | 25/16 | 13/8 | 27/16 | 7/4 | 29/16 | 15/8 | 31/16 | 2 |
| At the 7/4 Frame | 1 | 67/64 | 35/32 | 73/64 | 19/16 | 79/64 | 41/22 | 85/64 | 11/8 | 91/64 | 47/22 | 97/64 | 25/11 | 103/64 | 53/32 | 109/64 | 7/4 |
| At the 3/2 Frame | 1 | 33/32 | 17/16 | 35/32 | 9/8 | 37/32 | 19/16 | 39/32 | 5/4 | 41/32 | 21/16 | 43/32 | 11/8 | 45/32 | 23/16 | 47/32 | 3/2 |

**Figure 1**. *Graphic representation of abstract/empirical ratio correspondences at the $^3/_2$, $^7/_4$, $^2/_1$, $^9/_4$, and $^5/_2$ frames.*

As an example, let us say that we wish to augment the interval, $^7/_4$ (a small minor seventh), by projecting it onto a frame larger that $^2/_1$. We select $^9/_4$, very nearly an equal tempered major ninth, as our frame, and proceed:

$$g_{9/4}(^7/_4) \quad = \quad ^{31}/_{16} \quad ,$$

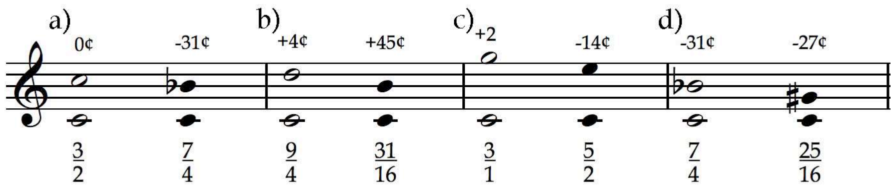
The ratio $^{31}/_{16}$ corresponds to the interval found between the 31st and 16th terms of the overtone series, and is somewhat less than an octave ($^{32}/_{16} = ^2/_1$). Its size in cents is approximately 1145¢. Had we selected $^3/_1$ as the frame (an interval very near an equal-tempered perfect twelfth), we find:

$$g_{3/1}(^7/_4) \quad = \quad ^5/_2 \quad ,$$

which gives an interval of an octave plus a just major third (≈ 1586¢). If we wish to undertake a diminution of $^7/_4$, we use a frame smaller than the octave. Let us use the frame $^7/_4$, itself:

$$g_{7/4}(^1/_1) \;=\; ^{25}/_{16} \quad ,$$

which is equivalent to two just major thirds ($^5/_4 \cdot {}^5/_4$) and is approximately 773¢, whereas our original interval, $^7/_4$, is approximately 969¢. These reframings of $^7/_4$ are presented in musical notation below (see Ex. 1).




**Example 1.** *Musical notation representing reframing of a $^7/_4$ frequency ratio at the a) $^2/_1$, b) $^9/_4$, c) $^3/_1$, and d) $^7/_4$ frames. Open noteheads indicate the frame, itself, filled noteheads indicate the reframed $^7/_4$. Frequency ratios for each are given below, cent deviations from equal temperament are given above each notated interval.*

When employing pitch augmentation or diminution, it will be necessary to define a reference pitch that will remain constant and against which the intervals are measured (the reference pitch, $^1/_1$ is not altered by reframing). Let us take augmentation and diminutions of a major triad in just intonation (Ex. 2), which is denoted (in root position) by the pitch ratios $^1/_1$, $^5/_4$, $^3/_2$. To find the empirical form of the major triad abstraction in the $^3/_1$ frame, we first assign the given values to their relevant variables: $f = {}^3/_1$, $a_1 = {}^1/_1$, $a_2 = {}^5/_4$, and $a_3 = {}^3/_2$. We can then calculate the empirical ratios for the major triad abstraction in the $^3/_1$ frame:

$$g_{3/1}(^1/_1) \;=\; ^1/_1 \qquad (= 0¢)$$

$$g_{3/1}(^5/_4) \;=\; ^3/_2 \qquad (\approx 702¢)$$

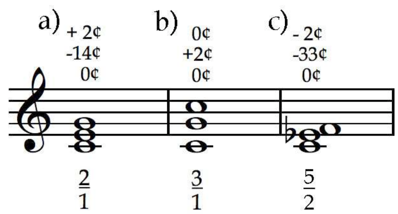
$$g_{3/1}(^3/_2) \;=\; ^2/_1 \qquad (1200¢).$$

The $^1/_1$ abstraction remains unchanged (as will always be the case), the $^5/_4$ abstraction in the $^3/_1$ frame becomes the empirical ratio $^3/_2$, and the $^3/_2$ abstraction in the $^3/_1$ frame becomes the empirical ratio $^2/_1$. Similarly, we find the empirical form of the major triad abstraction in the $^5/_3$ frame (a just major sixth):

$$g_{5/3}(^1/_1) = {}^1/_1 \quad (= 0¢)$$

$$g_{5/3}(^5/_4) = {}^7/_6 \quad (\approx 267¢)$$

$$g_{5/3}(^3/_2) = {}^4/_3 \quad (\approx 498¢) \quad .$$




**Example 2.** *Musical notation representing a root position, close position triad at the a) $^2/_1$, b) $^3/_1$, and c) $^5/_2$ frames.*

This process will work, even where some $a$ – 1 is less than one, so long as it is above a certain threshold determined by the frame. Positive values for empirical ratios result from all values of $a$ where $(a\text{-}1)(f\text{-}1) + 1 > 0$. The threshold occurs, therefore, at:

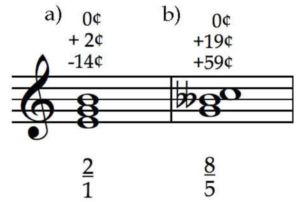
$$a = {}^{-1}/_{(f-1)} + 1 \quad ,$$

which results in an empirical frequency ratio (remember that this ratio is a multiplier to the original frequency) of zero (i.e. the $y$–intercept of the reframing function). For all values of $a < {}^{-1}/_{(f-1)} + 1$, the resulting empirical ratios will be negative values. Let us take, as an example, a first inversion, close position triad with $^1/_1$ at the top (Ex. 3), which would be expressed (at the $^2/_1$ frame) as: $^5/_8$, $^3/_4$, $^1/_1$. Diminishing the structure at the $^8/_5$ (just minor sixth) frame, we get:

$$g_{8/5}(^5/_8) = {}^{31}/_{40} \quad (\approx -441¢)$$

$$g_{8/5}(^3/_4) = {}^{17}/_{20} \quad (\approx -281¢)$$

$$g_{8/5}(^1/_1) = {}^1/_1 \quad (0¢) \quad .$$




**Example 3.** *Musical notation representing a first inversion, close position triad at the a) $^{2}/_{1}$ and b) $^{8}/_{5}$ frames.*

Beyond, simply creating reframed ratio sets from a set of shared abstract ratios, it may be useful to identify points of intersection between two frames, which is accomplished by finding those abstract ratios for each frame that will produce the same empirical ratio. To find the abstract ratio correlated to some empirical ratio at a given frame, we use the inverse function of $g_f(\text{a})$, $h_f(e)$:

$$h_f(e) = [(e - 1) \div (f - 1)] + 1 \quad .$$

For instance, to determine the abstract ratio of the empirical ratio $^{14}/_{11}$ in the $^{7}/_{2}$ frame:

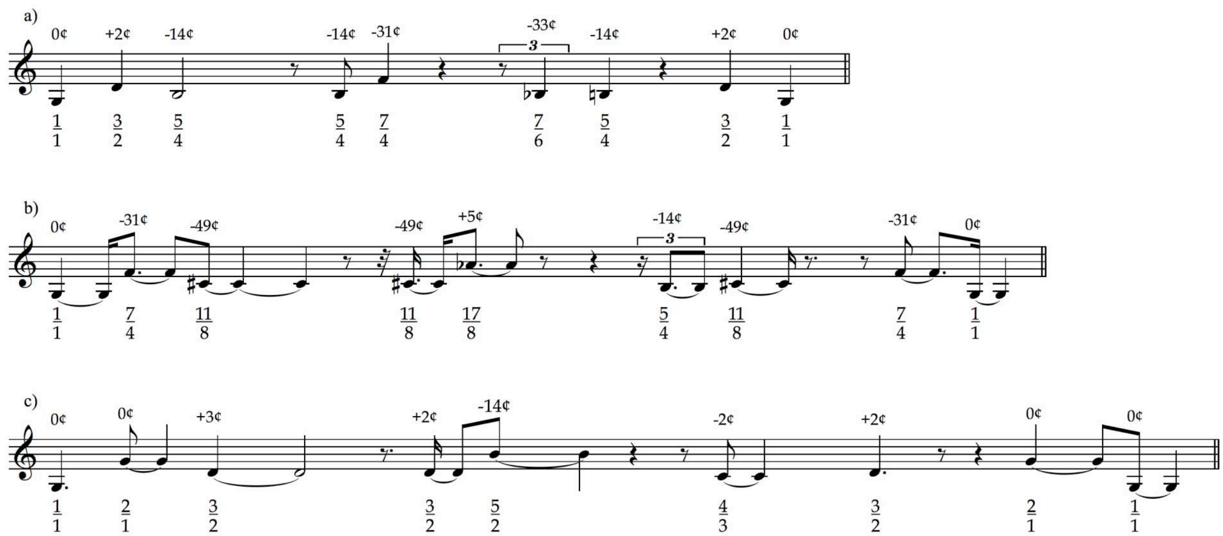
$$h_{7/2}(^{14}/_{11}) \quad = \quad ^{61}/_{55} \quad .$$

Given a set of empirical pitch ratios, we can find the points of perceptual intersection (at the empirical pitch ratios) across different frames. With the introduction of some criterion, we can have a justification for preferring one interpretation of the empirical ratios, as being certain abstract ratios in some frame, to all others. Such a criterion will be introduced in section 2.2., below; first, however, we shall present an explanation of transposition and inversion of reframed ratio sets.

### 2.1. Musical Operations under Reframing: Transposition and Inversion

As discussed thus far, we see that reframing can be used as a means of varying pitch and intervallic material in the creation of a composition in extended just intonation.

The most direct and perceivable application of reframing will be to linear contexts where contour identity is to be retained, whether in sequential melodic or in overlapping polyphonic statements. Example 4 presents such a case: a partial phrase taken from the author's string trio, *Cognitive Congruence*, shown in a) the original pitch content and rhythmic values, b) pitch and rhythmic content augmented at the proportion of $^{5}/_{2}$ (the $^{5}/_{2}$ frame, with respect to the pitch material), and c) pitch and rhythmic content augmented at the proportion of $^{3}/_{1}$ (the $^{3}/_{1}$ frame, with respect to the pitch material).




**Example 4:** *A musical line at the a) 2/1 , b) 5/2 , and c) 3/1 frames, with corresponding rhythmic augmentation as found in Douglas C. Wadle's* Cognitive Congruence. *Pitch ratios are given below each notated pitch, cent value deviations from equal temperament are given above each notated pitch.*

In *Cognitive Congruence*, the phrases from which the above material is drawn are part of a compound line. Each compound line is presented independently, in mensural cannon with each of the other compound lines, and as a mensural cannon with all three compound lines. This contrapuntal use of reframed musical lines raises the issue of the application of reframing to other musical operations derived from contrapuntal practice (i.e. reframed ratio sets under transposition and inversion).

To this end, a brief remedial discussion of transposition and inversion of frequency ratios is in order. Remembering that we are, in basic reframing, only concerned with pitch and pitch interval ratios (not pitch class or interval class ratios), we can see that any frequency ratio $p/q$ can be transposed by any other interval $r/s$ simply by multiplying. As the second ratio represents a pitch interval by which the first ratio is to be transposed, ascending intervals will be represented by ratios greater than one. The corresponding descending intervals will fall between zero and one and are the inverse of their ascending counterparts (e.g. $3/2$ is an ascending perfect fifth, $2/3$ a descending one). So for transposition of some frequency ratio $p/q$ at some other ratio $r/s$, we use:

$$T_{r/s}(p/q) = (p/q)(r/s) \quad .$$

Inversion around some axis $m/n$ is accomplished by dividing the axis by the original ratio, and then multiplying that result by the axis of inversion:

$$(m/n)[(m/n) \div (p/q)] = (m/n)[(m/n)(q/p)] = (m/n)^2(q/p) \quad .$$

In the case of an inversion around the axis of $1/1$, this amounts to simply inverting the numerator and denominator of the original ratio. For other axes, it amounts to finding the difference between the ratio to be inverted and the axis of inversion, and then transposing, from the axis, by the inverse of this difference. These situations can be expressed more simply as a combined transposition and inversion operation:

$$T_{r/s}I(p/q) = (r/s) \div (p/q) = (r/s)\,(q/p) \ ,$$

where $(^r/_s) = (^m/_n)^2$; this is tantamount to inverting around the axis of $^1/_1$, and then transposing by $(^r/_s)$.

With this in mind, let us compare the results we found above (Ex. 3) for a first inversion triad at the $^8/_5$ frame, with $^1/_1$ at the top, to the results we get if we find the empirical ratios of the first inversion major triad abstraction at the $^8/_5$ frame with $^2/_1$ at the top, consisting of the abstract ratios $^5/_4$, $^3/_2$, $^2/_1$ :

$$g_{8/5}(^5/_4) = {}^{23}/_{20} \quad (\approx 242¢)$$

$$g_{8/5}(^3/_2) = {}^{13}/_{10} \quad (\approx 454¢)$$

$$g_{8/5}(^2/_1) = {}^{8}/_{5} \quad (\approx 814¢) \quad .$$

The difference between corresponding ratios in these two sets is $e_1 = {}^{46}/_{31}$ (≈ 683¢), $e_2 = {}^{26}/_{17}$ (≈ 736¢), $e_3 = {}^8/_5$ (≈ 814¢). This alerts us to an anomaly that creeps into the handling of frequency ratios when one tries to transpose musical intervals using two (or more) reframed ratios (in frames other than $^2/_1$). We might expect that the results of our first inversion triad with the $^1/_1$ abstraction at its top should have all elements $^8/_5$ apart from the first inversion triad with the $^2/_1$ abstraction at its top, where the reframed octave is $^8/_5$, but this is clearly not the case. Had it been, we would have a value for the empirical ratio of the $^5/_4$ abstraction in the $^8/_5$ frame of

$$^{31}/_{40} \cdot {}^8/_5 = {}^{31}/_{25} \quad (\approx 372¢)$$

rather than $^{23}/_{20}$. This shows that for all frames where empirical ratios and abstract ratios differ (frames other than two or one[4]), simply multiplying empirical ratios resulting from the reframing function will not lead to the proper empirical ratio of the

[4] The case of the $^1/_1$ frame transforms all ratios in the set to $^1/_1$ as, $^1/_1 - 1 = 0$, and so any reframed interval will be:
$0 \cdot {}^a/_b + 1 = 1 = {}^1/_1$ .

corresponding abstract ratio found according to that function[5]. Where one of the empirical ratios remains constant, we will see a line that intersects the line defined by the framing function at the given frame at the point of that empirical ratio. The slope of this line can be determined by multiplying the consistent ratio by the empirical ratio of the correlated abstract ratio's inverse (to get the distorted $^1/_1$ abstraction), and its octave complement (to get the distorted $^2/_1$ abstraction), dividing the smaller into the greater and subtracting one[6]. For example, the distortions resulting from multiplying the reframed $^5/_2$ abstraction will lead to a line of slope $^{84}/_{63}$, a slightly steeper grade than the pure $^9/_4$ frame's slope of $^5/_4$ (see Fig. 2).

This discrepancy can be corrected by introducing the following factor into the multiplication of reframed ratios, where two ratios, $^p/_q$ and $^r/_s$, stand for empirical ratios connected to abstractions within a common frame, represented by the ratio $^x/_y$:

$$1 - \{[(p - q)(r - s)][(^{x - 2y}/_{x - y}) \div pr]\} \quad .$$

So, in order to make the product of the empirical ratios of the $^5/_8$ and $^2/_1$ abstractions in the $^8/_5$ frame (i.e. $g_{8/5}(^5/_8) \cdot g_{8/5}(^2/_1)$) correspond to the empirical ratio of the abstraction of the product of $^5/_8$ and $^2/_1$ (i.e. $g_{8/5}[(^5/_8)(^2/_1)]$), we proceed:

$$(^{31}/_{40} \cdot {}^8/_5)\{1 - [(31 - 40)(8 - 5)][(^{8 - (2 \cdot 5)}/_{8 - 5}) \div (31 \cdot 8)]\} = {}^{23}/_{20} \quad ,$$

which is precisely what we found when simply reframing the ratio, $^5/_4$, in the $^8/_5$ frame. Returning to our transposition function, we get a generalization of:

$$T_{r/s}(^p/_q) = (^p/_q)(^r/_s)(1 - \{[(p - q)(r - s)][(^{x - 2y}/_{x - y}) \div pr]\})$$

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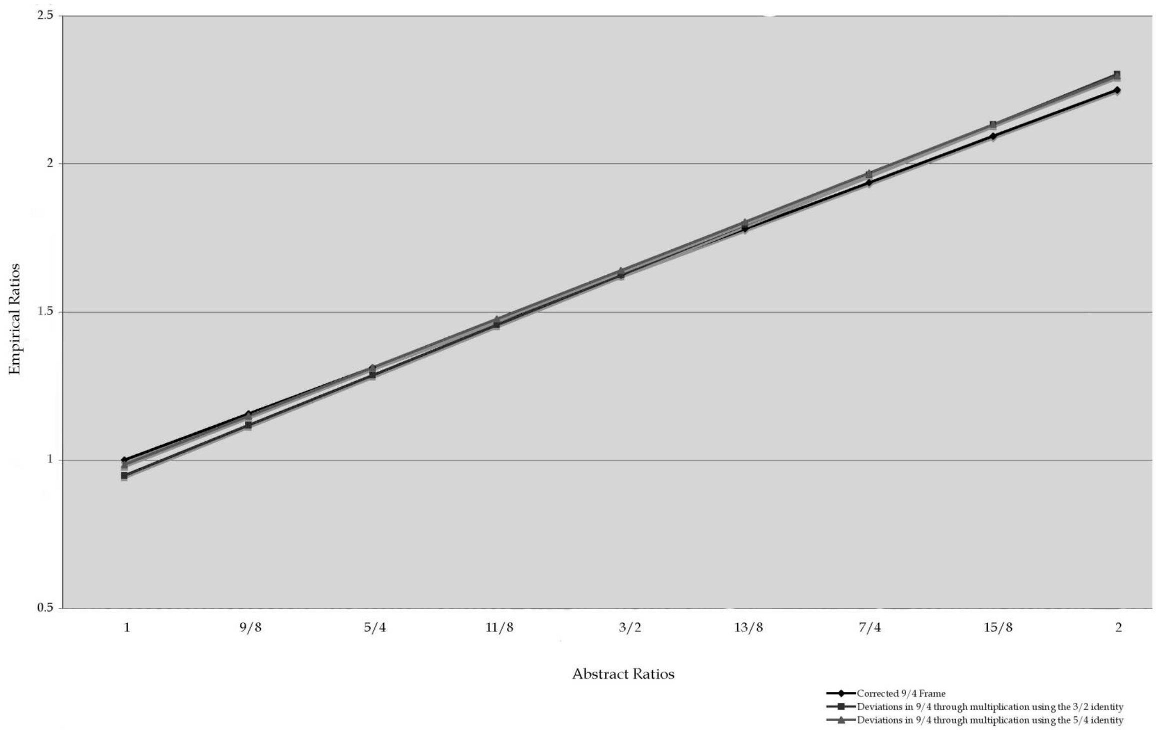
[5] In fact, for all frames other than $^2/_1$, the reframed octave, through successive ascending iterations, approaches 2 such that: $\lim_{n\to\infty}[g_f(2^n) \div g_f(2^{n-1})] = 2$, for all $f \neq 2$.

[6] Note that this will result in a line parallel to that of the undistorted frame of the associated slope.

for all frames, $^{x}/_{y}$. Inversion operations are somewhat more problematic. The simplest way to proceed is to find the empirical ratio, $^{w}/_{v}$, for the inverse of the abstract ratio, $^{m}/_{n}$, associated with $^{p}/_{q}$ such that $g_{x/y}(^{m}/_{n}) = {}^{a}/_{b}$ and $h_{x/y}(^{p}/_{q}) = {}^{m}/_{n}$, by the reframing function $g_{x/y}(^{n}/_{m}) = {}^{w}/_{v}$. This will give the inversion around the axis of $^{1}/_{1}$. We can then simply transpose as above for any value of $^{r}/_{s}$ using the corrective factor for the product of two reframed ratios.




**Abstract Ratios:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| At Corrected 9/4 Frame | 1<br>0¢ | 37/32<br>251¢ | *21/16*<br>471¢ | 47/32<br>666¢ | *13/8*<br>841¢ | 57/32<br>999¢ | 31/16<br>1145¢ | 67/32<br>1279¢ | 9/4<br>1404¢ |
| Using 3/2 Abstraction of 9/4 Frame | 91/96<br>-93¢ | 143/128<br>192¢ | 247/192<br>436¢ | 559/384<br>650¢ | *13/8*<br>841¢ | 689/384<br>1012¢ | 377/192<br>1168¢ | *273/128*<br>1311¢ | 221/96<br>1444¢ |
| Using 5/4 Abstraction of 9/4 Frame | 63/64<br>-27¢ | 147/128<br>240¢ | *21/16*<br>471¢ | 189/128<br>675¢ | 105/64<br>857¢ | 231/128<br>1022¢ | 63/32<br>1173¢ | *273/128*<br>1311¢ | 147/64<br>1439¢ |

**Figure 2.** *Graphic representation of distortion due to compounding of reframed frequency ratios at the $^{9}/_{4}$ frame.*

Having presented the mechanics behind some musical applications of reframed frequency ratio sets at the local level, we are now in a position to approach broader theoretical questions regarding the superset of reframed frequency ratio sets, and certain perceptual and theoretical constraints to which we may wish to subject frequency ratios sets.

## 2.2. Finding the Prime Form of a Given Frame (The Superset of Transformed Ratio Sets)

When dealing with multiple empirical ratio sets based on the same set of abstract ratios (and vice versa), whether it construed as intervallic structures independently of any reference point, or to transform pitch collections defined by reference to a given $^{1}/_{1}$, it will be useful to have a naming convention by which we indicate that superset. Borrowing a page from set theory, we can propose a prime form of the membership of that superset to serve as the designator for all members of that superset. The primary frame will be determined by finding the most compact form of the frequency ratio set structure using a method, to be presented below, that adapts Tenney's principle of "crystal growth" (Tenney 2008). The procedure involves the measurement of harmonic distance between two pitch ratios, defined by Tenney according to the formula:

$$\log_2(m \cdot n) \quad ,$$

where the ratio $^{m}/_{n}$ expresses the interval ratio between the two pitches. According to Tenney, the most compact ratio set[7] will be that with the lowest harmonic distance

[7] Tenney developed the concept of "crystal growth" in reference to frequency ratio lattices of the sort first proposed by Leonhard Euler (1739) and extended by Tenney (1984) to include higher primes and to return functionally useful information about relative consonance and dissonance relations between pitches within the lattice. In lattice-structure pitch space, every prime number allowed in a tuning system governs one dimension of harmonic space. The concept of crystal growth according to HD sum does not, however, require a closed lattice to be intelligible and so we may apply it without broaching the subject of lattices, proper (which will be saved for a later paper in this series).

(HD) sum; that is, the set with the smallest value where the harmonic distance from every pitch in the set to every other pitch in the set is measured as an HD value and added together. I will further follow the suggestion of Wolfgang von Schweinitz that, where crystals have the same HD sum, the simpler ratio is preferred (i.e. the ratio with the smaller HD value measured from $^1/_1$) (Sabat 2008). If there is still a tie, a random selection is made. For our purposes, when two frames are found to have equally compact HD sums, we can state that it has dual primary forms. We remove the $^1/_1$ frame as a candidate for the primary frame, as it results in a set of empirical ratios all of which equal $^1/_1$ , for a composite HD sum of zero:

$$g_{1/1}(^m/_n) = {}^1/_1 \text{, for all } ^m/_n \quad .$$

The following is a simple procedure for the determination of the prime frame of a given set of empirical frequency ratios:

1.) Given a set $S$ of empirical ratios: $\{p_1, p_2, p_3, p_4, \ldots p_n\}$, find the set of ratios, $T$: $\{q_1, q_2, q_3, q_4, \ldots q_n\} = \{p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1, \ldots p_n - 1\}$.
2.) Find the prime factors for all numerators and denominators in the set $T$, and arrange them in a table (as in Table 1) where the rows correspond to a single appearance of a factor, each factor having a number of rows equal to the greatest number of times it appears within the set of numbers under consideration (numerators or denominators of $T$).
3.) For each row of the table in which half or more of the respective numerators share the prime factor for that row, include that factor as a factor determining the numerator of the ratio $f$ - 1. Do the same for the factors of

denominators appearing in $T$ in order to find the denominator of $f$ - 1.

Express $f$ - 1 as a fraction in lowest terms.

4.) Add one to the ratio $f$ – 1 to find $f$, which is the primary frame.

For example, given a set of empirical ratios, $S$: {$^1/_1$, $^9/_8$, $^7/_6$, $^5/_4$, $^{11}/_8$, $^3/_2$, $^{13}/_8$}, we calculate the set of related ratios, $T$: {0, $^1/_8$, $^1/_6$, $^1/_4$, $^3/_8$, $^1/_2$, $^5/_8$}. We very quickly see that the only factor appearing in the numerator of more than half the members of the set, $T$, of $q$'s is one, so this will be our numerator. We look to the factorization of denominators in $T$ to see if any appear more that three times. Referring to Table 1, we see two appearing in more than half the instances of the denominators contained in our first and second rows and no other values occurring in more than half the instances of the denominators of $T$, which tells us that $2^2 = 4$ will form our target denominator for $f$ - 1. This gives a value of $^1/_4$. We then find $f = {}^1/_4 + 1 = {}^5/_4$, and so the prime frame for the given perceived pitch set $S$ is $^5/_4$. If we wish to check the reasonability of this finding, we can calculate the abstract ratios correlated to the empirical ratio set $S$ at the $^5/_4$ frame by the function $h_f(a)$. This results in a set of abstract ratios at the $^5/_4$ frame, $R$: {$^1/_1$, $^3/_2$, $^5/_3$, $^2/_1$, $^5/_2$, $^3/_1$, $^7/_2$}, which, intuitively, looks quite a bit simpler than $S$. We can verify by the construction of a matrix detailing the relations between all the pitches within each set, calculating the HD sum for each and comparing. If we do so, we will find an HD sum of approximately 138.6 for $S$ and a value of approximately 81.4 for $R$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 8 | 6 | 4 | 8 | 2 | 8 |
| | 2 | 2 | 2 | 2 | 2 | 2 |
| | 2 | | 2 | 2 | | 2 |
| | 2 | | | 2 | | 2 |
| | | 3 | | | | |

**Table 1**. *Arrangement of the prime factors of denominators in the set*, T: {*0*, ${}^{1}/_{8}$, ${}^{1}/_{6}$, ${}^{1}/_{4}$, ${}^{3}/_{8}$, ${}^{1}/_{2}$, ${}^{5}/_{8}$} *to aid in the determination of the primary frame for the empirical pitch ratio set*, S: {${}^{1}/_{1}$, ${}^{9}/_{8}$, ${}^{7}/_{6}$, ${}^{5}/_{4}$, ${}^{11}/_{8}$, ${}^{3}/_{2}$, ${}^{13}/_{8}$}.

With this means of referring to variable sets of ratios linked by virtue of some reframing relation, we can identify supersets of ratios (akin to set theory's set classes) and, by extension, identify the musical correspondences that obtain via these reframing transformations as discussed in the preceding sections. The reframing of frequency ratio sets may be seen to have deeper conceptual implications, as well. We will not tackle these implications directly in this paper, but we will introduce one additional consideration that will complete the foundation for undertaking these considerations in the future: the determination of cognized ratio sets from a set of empirical ratios.

**3.0. Cognized Ratio Sets**

Concerning perception, we will have recourse to what Tenney has referred to as "tolerance". Tolerance is our capacity to interpret complex frequency relations as their simplest nearest neighbor (Belet and Tenney 1987, Tenney 2001; see also Bregman 1990 and Terhardt 1974 on the related phenomenon of spectral fusion and Vos 1982 and 1984 on beats and pitch discrimination). A standard example is the equal tempered major third vs. the just, ${}^{5}/_{4}$, "major third", which have a discrepancy of about 14¢. The principle of tolerance states that, in hearing the irrational relationship of the equal tempered third (in isolation), our perceptual apparatus ascribes the meaning associated with the just interval. This being stated, we can see that, in attempting to arrive at interpretations of a frequency ratio, we must subject the mathematical exactitude to the vagaries of perception in the form of tolerance. To this end, empirical ratios will be reinterpreted as "cognized" ratios (designated by "*c*") within some pre-established tolerance threshold. For instance, if we allow a tolerance range of 10¢ in either direction from the empirical ratio, $^{35}/_{16}$ , which has a cent value of approximately 1355.1, we can find the ratios $^{24}/_{11}$ (≈ 1350.6¢) and $^{11}/_{5}$ (≈ 1365.0¢), both within the tolerance range. The harmonic distance of $^{24}/_{11}$ is $\log_2(24 \cdot 11) \approx 8.04$, whereas the harmonic distance of $^{11}/_{5} \approx 5.78$ and that of $^{35}/_{16} \approx 9.13$. Clearly, the preferred ratio within the specified tolerance range will be $^{11}/_{5}$ as the cognized ratio of the empirical ratio $^{35}/_{16}$ .

The determination of the preferred cognized frequency ratio in a musical context (rather than as an isolated interval) will be dependent upon the empirical frequency ratios then sounding in comparison with available interpretations of all empirical ratios sounding in the piece (or some analytically pertinent segment of the piece) and a specified tolerance range. In works to date, I have used a procedure in which a tolerance range is defined (e.g. 10¢) and all empirical ratios occurring in the work are compared with ratios found on a lookup table[8] falling within that tolerance range. The value selected is that which will lead to the simplest set of harmonic

[8] An extensive lookup table, running upwards of 400 pages, is available from the author as a spreadsheet that can be filtered in a number of ways to reduce the number of candidate values to a manageable number.

relations throughout the set of ratios (i.e. the lowest HD sum). The total resulting set of ratios is, then, the set of cognized ratios[9].

An additional wrinkle may be added to the network of abstract, empirical, and cognized ratio sets – namely, the imposition of a limitation of the highest prime factor allowed in the elements of the ratio set of any or all of the types discussed. The reasons one might have for introducing such a limit are to ensure playability – ratios with higher prime terms are generally harder to tune than those with lower prime terms – or to restrict dimensionality at some level of metaphoric significance when working with lattice structures. This will introduce yet another form of the frequency ratio set for consideration in reference to the others. This we call the "restricted cognized set at the *n*-limit". To effect such a limiting, the algorithm for determining cognized ratio sets employs a lookup table filtered to omit all frequency ratios that do not fall within the specified prime limit.

**4.0. Conclusion**

For the composer interested in perception and the apprehension of form, reframing can be used to explore the relation of theoretical frequency ratio constructs (abstract ratio sets) to empirically present ratios (what is present in the physical world, empirical ratio sets) to conceivable form (cognized ratio sets). Compositional work in this realm

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[9] I am, at present, at work on a tolerance algorithm for determining the cognized ratios from empirical ratios that allows variable inputs for tolerance range, highest partial used in fine-tuning, least number of acoustic beats required for fine-tuning, duration of tunable intervals, and allowable pitch-center drift in reference to a just noticeable difference across temporal gestalt boundaries. Such an algorithm should not only be a fantastically useful tool for the purposes described herein, but also for the analysis of music in tempered systems with respect to its harmonic (in the mathematical sense) implications.

assumes the validity of articulating pitch structures that cannot be heard as evincing these structures (hence, these uses place an additional weight upon the information conveyed within the score where a score is used). While the validity of such an approach is not widely accepted, I think it is both defensible and interesting, but I will save the defense for another time. Applications, of the material contained herein (particularly in section 3.0.), for electronic instruments may be found without relying on such an approach[10].

This paper has focused only on the mechanics of intervallic augmentation and diminution as a transformation of extended just intonation frequency ratios, along with naming conventions for these transformations and their products. Some small attention has been paid to the application of these tools to musical composition and certain musical operations under reframing. It remains for future papers to examine particular uses of reframing with existing compositions; the implications of each form of ratio set discussed herein for musical cognition and perceptual experience, more generally; and, finally, the projection of these ratio sets onto the lattice structures of standard extended just intonation theory. This excursion into speculative music theory should be understood as neither the first shot fired in a struggle to establish a system – as in the development of serial practices – nor as an attempt to introduce a new analytical paradigm – as in set theory – that will provide us with tools to analyze preexisting works. I am aiming, merely, at the explication of new (conceptually based)

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[10] See Marc Sabat's "Algorithm for Real-Time Harmonic Microtuning", text from a paper presented at the Congress on Sound and Music Computing (2008) available on his page at Plainsound Music Edition (www.plainsound.org), for a description of related research initiated in collaboration with me in 2007 at the California Institute of the Arts, though we subsequently pursued this work independently of one another.

tools for application in musical composition with a description of certain applications of those tools that have occurred to me. Any use of this material, and any extension thereof, that seems, to the one undertaking it, worth the undertaking, is encouraged.

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